# On periodic Takahashi manifolds*

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**Abstract**

In this paper we show that periodic Takahashi 3-manifolds are cyclic coverings of the connected sum of two lens spaces (possibly cyclic coverings of $\mathbf{S}^3$), branched over knots. When the base space is a 3-sphere, we prove that the associated branching set is a two-bridge knot of genus one, and we determine its type. Moreover, a geometric cyclic presentation for the fundamental groups of these manifolds is obtained in several interesting cases, including the ones corresponding to the branched cyclic coverings of $\mathbf{S}^3$.

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## 1 Introduction

Takahashi manifolds are closed orientable 3-manifolds introduced in [21] by Dehn surgery on $\mathbf{S}^3$, with rational coefficients, along the $2n$-component link $\mathcal{L}_{2n}$ depicted in Figure 1. These manifolds have been intensively studied in [11], [19], and [22]. In the latter two papers, a nice topological characterization of all Takahashi manifolds as two-fold coverings of $\mathbf{S}^3$, branched over the closure of certain rational 3-string braids, is given.

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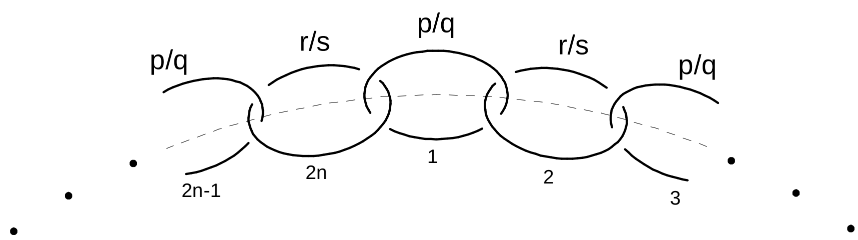


Figure 1: Surgery along $\mathcal{L}_{2n}$ yielding $M_n(p/q, r/s)$.

A Takahashi manifold is called *periodic* when the surgery coefficients have the same cyclic symmetry of order $n$ of the link $\mathcal{L}_{2n}$, i.e. the coefficients are $p/q$ and $r/s$ alternately. Several important classes of 3-manifolds, such as (fractional) Fibonacci manifolds [7, 22] and Sieradsky manifolds [2, 20], represent notable examples of periodic Takahashi manifolds.

In this paper we show that each periodic Takahashi manifold is an $n$-fold cyclic covering of the connected sum of two lens spaces, branched over a knot. This knot arises from a component of the Borromean rings, by performing a surgery with coefficients $p/q$ and $r/s$ along the other two components.

For particular values of the surgery coefficients (including the classes of manifolds cited above), the periodic Takahashi manifolds turn out to be $n$-fold cyclic coverings of $\mathbf{S}^3$, branched over two-bridge knots of genus one[1], whose parameters are obtained using Kirby-Rolfsen calculus [18] (compare the analogous result of [11], obtained by a different approach). Observe that in [19] a characterization of all periodic Takahashi manifolds as $n$-fold cyclic coverings of $\mathbf{S}^3$, branched over the closure of certain rational 3-string braids, is presented, but the result is incorrect, as we show in Remark 1.

For many interesting periodic Takahashi manifolds - including the ones corresponding to branched cyclic coverings of $\mathbf{S}^3$ - a cyclic presentation for the fundamental group is provided and proved to be geometric, i.e. arising from a Heegaard diagram, or, equivalently, from a canonical spine[2] [16].

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[1]For notation and properties about two-bridge knots and links we refer to [1]. For the characterization of two-bridge knots of genus one, see [5].

[2]A canonical spine is a 2-dimensional cell complex with a single vertex.

# 2 Main results

We denote by $M(p_1/q_1,\dots,p_n/q_n;r_1/s_1,\dots,r_n/s_n)$ the Takahashi manifold obtained by Dehn surgery on $\mathbf{S}^3$ along the $2n$-component link $\mathcal{L}_{2n}$ of Figure 1, with surgery coefficients $p_1/q_1,r_1/s_1,\dots,p_n/q_n,r_n/s_n \in \widetilde{\mathbf{Q}} = \mathbf{Q}\cup\{\infty\}$ respectively, cyclically associated to the components of $\mathcal{L}_{2n}$.

A Takahashi manifold is periodic when $p_i/q_i = p/q$ and $r_i/s_i = r/s$, for every $i=1,\dots,n$. Denote by $M_n(p/q,r/s)$ the periodic Takahashi manifold $M(p/q,\dots,p/q;r/s,\dots,r/s)$. From now on, without loss of generality, we can always suppose that: $\gcd(p,q)=1$, $\gcd(r,s)=1$ and $p,r\geq 0$. Moreover, if $\alpha,\beta\in\mathbf{Z}$ with $\alpha\geq 0$ and $\gcd(\alpha,\beta)=1$, we shall denote by $L(\alpha,\beta)$ the lens space of type $(\alpha,\beta)$. As usual, $L(0,1)$ is homeomorphic to $\mathbf{S}^1\times\mathbf{S}^2$ and $L(1,\beta)$ is homeomorphic to $\mathbf{S}^3$, for all $\beta$ (including $\beta=0$).

Notice that $M_n(p/q,-p/q)$ is the Fractional Fibonacci manifold $M_n^{p/q}$ defined in [22] and, in particular, $M_n(1,-1)$ is the Fibonacci manifold $M_n$ studied in [7]. Moreover, $M_n(1,1)$ is the Sieradsky manifold $M_n$ introduced in [20] and studied in [2]. Because of the symmetries of $\mathcal{L}_{2n}$, the homeomorphisms

$$M_n(p/q,r/s)\cong M_n(-p/q,-r/s)\cong M_n(r/s,p/q)\cong M_n(-r/s,-p/q)$$

can easily be obtained for all $n\geq 1$ and $p/q,r/s\in\widetilde{\mathbf{Q}}$.

A balanced presentation of the fundamental group of every Takahashi manifold is given in [21], and in [19] it is shown that this presentation is geometric, i.e. it arises from a Heegaard diagram (or, equivalently, from a canonical spine). As a consequence, $\pi_1(M_n(p/q,r/s))$ admits the following geometric presentation with $2n$ generators and $2n$ relators:

$$< x_1,\dots,x_{2n}\,|\, x_{2i-1}^q x_{2i}^{-r} x_{2i+1}^{-q},\ x_{2i}^s x_{2i+1}^p x_{2i+2}^{-s}\,;\ i=1,\dots,n >,$$

where the subscripts are mod $2n$.

When $r=1$, we can easily get a cyclic presentation [9] with $n$ generators:[3]

$$\pi_1(M_n(p/q,1/s)) = < z_1,\dots,z_n\,|\, z_i^p(z_i^{-q}z_{i+1}^q)^s(z_i^{-q}z_{i-1}^q)^s\,;\ i=1,\dots,n >, \tag{1}$$

where the subscripts are mod $n$.

**Proposition 1** *For all $p/q\in\widetilde{\mathbf{Q}}$ and $s\in\mathbf{Z}$, the cyclic presentation (1) of $\pi_1(M_n(p/q,1/s))$ is geometric.*

[3] Alternatively, a similar cyclic presentation can be obtained when $p=1$.

**Proof.** If $s=0$ then $M_n(p/q,1/s)$ is homeomorphic to the connected sum of $n$ copies of $L(p/q)$, and therefore the statement is straightforward. If $s>0$, the presentation becomes

$$< z_1,\ldots,z_n \,|\, z_i^{p-q}(z_{i+1}^q z_i^{-q})^s (z_{i-1}^q z_i^{-q})^{s-1} z_{i-1}^q \,;\, i=1,\ldots,n > . \qquad (1')$$

Figure 2 shows an RR-system which induces $(1')$, and so, by [17], this presentation is geometric. If $s<0$, the presentation becomes

$$< z_1,\ldots,z_n \,|\, z_i^{p+q}(z_{i+1}^{-q} z_i^{q})^{-s} (z_{i-1}^{-q} z_i^{q})^{-s-1} z_{i-1}^{-q} \,;\, i=1,\ldots,n > . \qquad (1'')$$

Therefore, if we replace $q$ with $-q$, Figure 2 also gives an RR-system inducing $(1'')$. ■

Since the link $\mathcal{L}_2$ is a two-component trivial link, we immediately get the following results:

**Lemma 2** *For all $p/q, r/s \in \widetilde{\mathbf{Q}}$, the manifold $M_1(p/q,r/s)$ is homeomorphic to the connected sum of lens spaces $L(p,q)\#L(r,s)$. In particular, $M_1(p/q,1/s)$ is homeomorphic to the lens space $L(p,q)$ and $M_1(1/q,1/s)$ is homeomorphic to $\mathbf{S}^3$.*

**Proof.** $M_1(p/q,r/s)$ is obtained by Dehn surgery on $\mathbf{S}^3$, with coefficients $p/q$ and $r/s$, along the trivial link with two components $\mathcal{L}_2$. ■

Now we prove the main result of the paper:

**Theorem 3** *For all $p/q, r/s \in \widetilde{\mathbf{Q}}$ and $n>1$, the periodic Takahashi manifold $M_n(p/q,r/s)$ is the $n$-fold cyclic covering of the connected sum of lens spaces $L(p,q)\#L(r,s)$, branched over a knot $K$ which does not depend on $n$. Moreover, $K$ arises from a component of the Borromean rings, by performing a surgery with coefficients $p/q$ and $r/s$ along the other two components.*

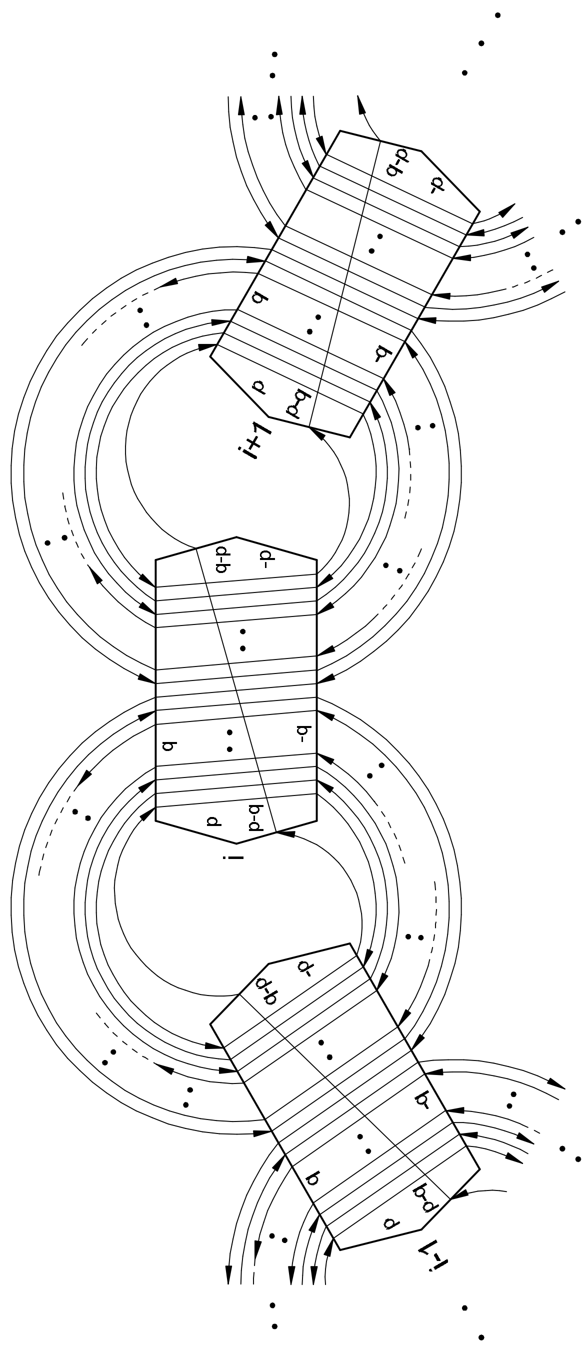


Figure 2: An RR-system for the cyclic presentation $(1')$.

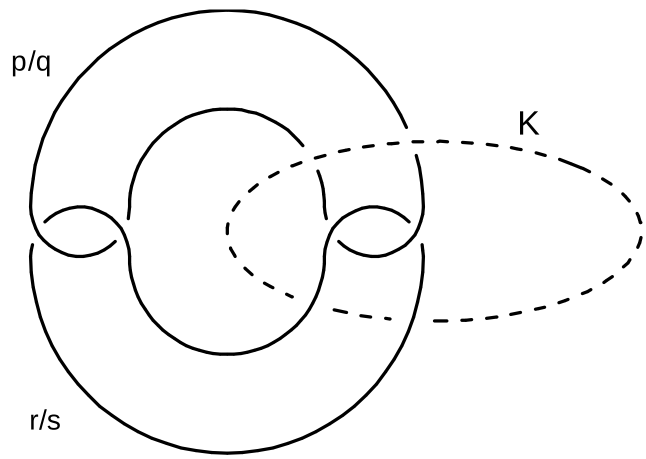


Figure 3: The branching set $K$ (dashed line).

**Proof.** Both the link $\mathcal{L}_{2n}$ and the surgery coefficients defining $M_n(p/q, r/s)$ are invariant with respect to the rotation $\rho_n$ of $\mathbf{S}^3$, which sends the $i$-th component of $\mathcal{L}_{2n}$ onto the $(i+2)$-th component (mod $2n$). Let $\mathcal{G}_n$ be the cyclic group of order $n$ generated by $\rho_n$. Observe that the fixed-point set of the action of $\mathcal{G}_n$ on $\mathbf{S}^3$ is a trivial knot disjoint from $\mathcal{L}_{2n}$. Therefore, we have an action of $\mathcal{G}_n$ on $M_n(p/q, r/s)$, with a knot $K_n$ as fixed-point set. The quotient $M_n(p/q, r/s)/\mathcal{G}_n$ is precisely the manifold $M_1(p/q, r/s)$, which is homeomorphic to $L(p, q)\#L(r, s)$ by Lemma 2, and $K_n/\mathcal{G}_n$ is obviously a knot $K \subset M_1(p/q, r/s)$, which only depends on $p/q$ and $r/s$. Moreover, $K \cup \mathcal{L}_2$ is the Borromean rings, as showed in Figure 3. This proves the statement. ■

We can give another description of the branching set $K$, as the inverse image of a trivial knot in a certain two-fold branched covering.

Denote by $\mathcal{L}(p/q, r/s)$ the link depicted in Figure 4. It is composed by the closure of the rational 3-string braid $\sigma_1^{p/q}\sigma_2^{r/s}$, which is the connected sum of the two-bridge knots or links $\mathbf{b}(p, q)$ and $\mathbf{b}(r, s)$, and by a trivial knot. Moreover, denote: (i) by $\mathcal{O}_n(p/q, r/s) = M_n(p/q, r/s)/\mathcal{G}_n$ the orbifold from the proof of Theorem 3, whose underlying space is $L(p, q)\#L(r, s)$ and whose singular set is the knot $K$, with index $n$; (ii) by $\mathbf{S}^3(\mathcal{K}_n(p/q, r/s))$ the orbifold whose underlying space is $\mathbf{S}^3$ and whose singular set is the closure of the rational 3-string braid $(\sigma_1^{p/q}\sigma_2^{r/s})^n$, with index 2; and (iii) by $\mathbf{S}^3(\mathcal{L}(p/q, r/s))$

the orbifold whose underlying space is $\mathbf{S}^3$ and whose singular set is the link $\mathcal{L}(p/q, r/s)$, with index 2 and $n$ as pointed out in Figure 4.

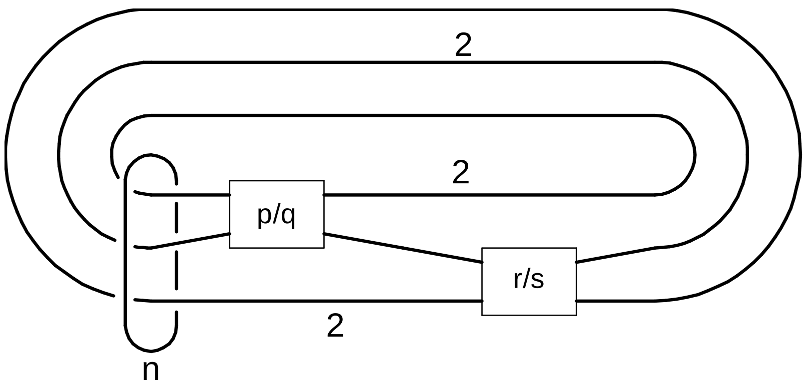


Figure 4: The link $\mathcal{L}(p/q, r/s)$.

**Proposition 4** *Assuming the previous notations, the following commutative diagram holds for each periodic Takahashi manifold.*

$$\begin{tikzcd}
 & M_n(p/q, r/s) \arrow[dl, "2"'] \arrow[dr, "n"] & \\
\mathbf{S}^3(\mathcal{K}_n(p/q, r/s)) \arrow[dr, "n"'] & & \mathcal{O}_n(p/q, r/s) \arrow[dl, "2"] \\
 & \mathbf{S}^3(\mathcal{L}(p/q, r/s)) &
\end{tikzcd}$$

**Proof.** The link $\mathcal{L}_{2n}$ admits an invertible involution $\tau$, whose axis intersects each component in two points (see the dashed line of Figure 1), and the rotation symmetry $\rho_n$ of order $n$ which was discussed in Theorem 3. These symmetries induce symmetries (also denoted by $\tau$ and $\rho_n$) on the periodic Takahashi manifold $M = M_n(p/q, r/s)$, such that $\langle \tau, \rho_n \rangle \cong \langle \tau \rangle \oplus \mathcal{G}_n \cong \mathbf{Z}_2 \oplus \mathbf{Z}_n$. We have $M/\langle \tau \rangle = \mathbf{S}^3(\mathcal{K}_n(p/q, r/s))$ (see [19] and [22]) and $M/\mathcal{G}_n = \mathcal{O}_n(p/q, r/s)$ (see Theorem 3). It is immediate to see that $\rho_n$ induces a symmetry (also denoted by $\rho_n$) on the orbifold $M/\langle \tau \rangle$, and $(M/\langle \tau \rangle)/\mathcal{G}_n$ is the orbifold $\mathbf{S}^3(\mathcal{L}(p/q, r/s))$. As we see from Figure 3, $\tau$

induces a strongly invertible involution (also denoted by $\tau$) on the link $\mathcal{L}_2$. Using the Montesinos algorithm we see that $(M/\mathcal{G}_n)/\langle\tau\rangle = \mathbf{S}^3(\mathcal{L}(p/q, r/s))$. This concludes the proof. ■

As a consequence, the branching set $K$ of Theorem 3 can be obtained as the inverse image of the trivial component of $\mathcal{L}(p/q, r/s)$ in the two-fold branched covering $\mathcal{O}_n(p/q, r/s) \to \mathbf{S}^3(\mathcal{L}(p/q, r/s))$.

From Theorem 3 we can get the following result, which has already been obtained in [11] by a different technique.

**Proposition 5** *For all $q, s \in \mathbf{Z}$ and $n > 1$, the periodic Takahashi manifold $M_n(1/q, 1/s)$ is the $n$-fold cyclic covering of $\mathbf{S}^3$, branched over the two-bridge knot of genus one $\mathbf{b}(|4sq-1|, 2s) \cong \mathbf{b}(|4sq-1|, 2q)$.*

**Proof.** From Theorem 3, $M_n(1/q, 1/s)$ is the $n$-fold cyclic covering of $L(1,q)\#L(1,s) \cong \mathbf{S}^3$, branched over a knot $K$ which does not depend on $n$. By isotopy and Kirby-Rolfsen moves it is easy to obtain (see Figure 5) a diagram of $K$, which is a Conway's normal form of type $[-2q, 2s]$. This proves the statement. ■

Proposition 5 covers the results of [2], [7] and [22] concerning $n$-fold branched cyclic coverings of two-bridge knots. Moreover, for all $p, q \in \mathbf{Z}$, the periodic Takahashi manifold $M_n(1/q, 1/s)$ is homeomorphic to the Lins-Mandel manifold $S(n, |4sq-1|, 2s, 1)$ [13, 15], the Minkus manifold $M_n(|4sq-1|, 2s)$ [14] and the Dunwoody manifold $M((|4q-1|-1)/2, 0, 1, s, n, -q_\sigma)$ [3, 6].

Moreover, observe that all cyclic coverings of two-bridge knots of genus one are periodic Takahashi manifolds.

**Remark 1** The results of Corollaries 8, 9 and 11 of [19], concerning periodic Takahashi manifolds as $n$-fold cyclic branched coverings of the closure of certain (rational) 3-string braids, are incorrect. This is evident from the following counterexamples. If $p/q = 3$ and $r/s = -3$ then the first homology group of the 3-fold cyclic branched covering of the closure of the 3-string braid $(\sigma_1^3\sigma_2^{-3})^2$ has order 256, but $|H_1(M_3(3,-3))| = 1296$. If $p/q = 3/2$ and $r/s = 1$ then the first homology group of the 4-fold cyclic branched covering of the closure of the rational 3-string braid $(\sigma_1^{3/2}\sigma_2)^2$ has order 135, but $|H_1(M_4(3/2, 1))| = 15$. Note that the corollaries are valid if $p = r = 1$.

The following conjecture is naturally suggested by the previous results.

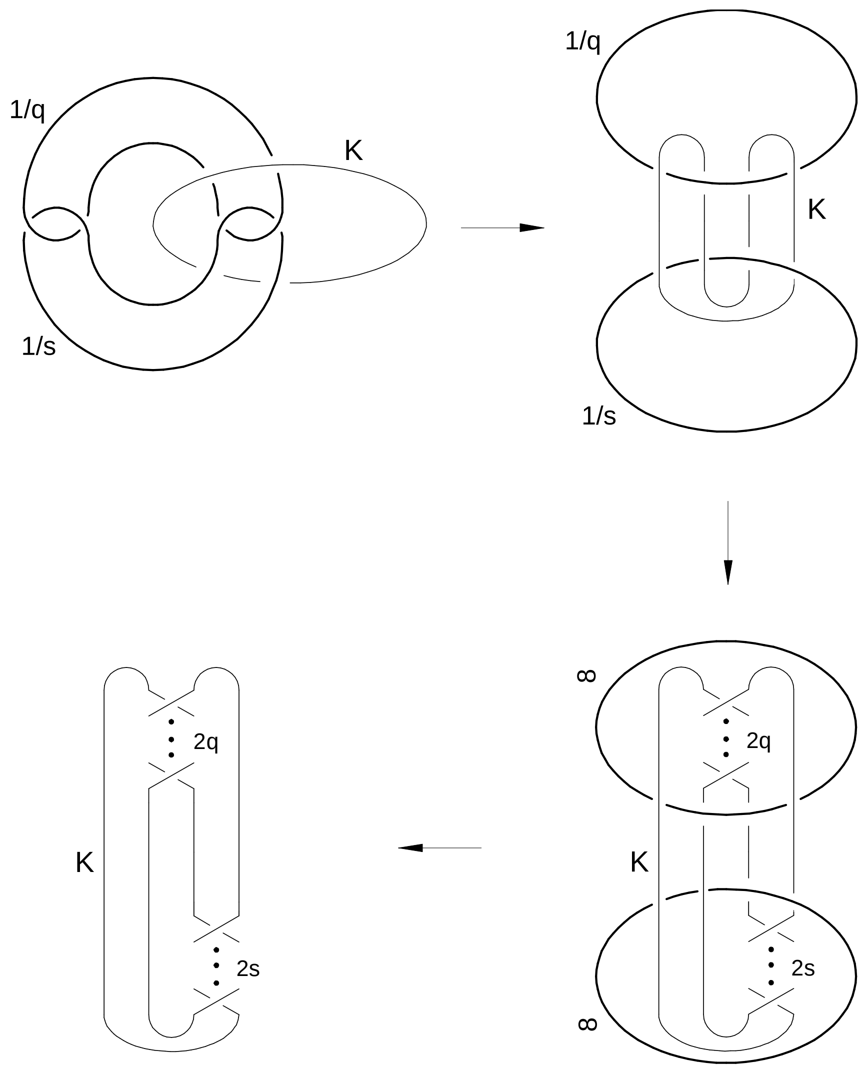


Figure 5:

**Conjecture** Let $p/q, r/s \in \widetilde{\mathbf{Q}}$ be fixed. Then, for all $n > 1$, the periodic Takahashi manifolds $T_n = M_n(p/q, r/s)$ are $n$-fold cyclic coverings of $\mathbf{S}^3$, branched over a knot which does not depend on $n$, if and only if $p = 1 = r$.

Added in revision - The referee pointed out that it is possible to prove the conjecture for "almost all cases" by using the hyperbolic Dehn surgery theorem and the shortest geodesic arguments by Kojima [12].

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